JOURNAL OF COMPUTATIONAL PHYSICS 73, 481–489 (1987)

Note

Transmittance of a Circular Aperture by an Integrable Fractional-Like Approximation to $J_0(x)$ Function

1. Introduction

A simple approximation to $J_0(x)$ function [1], based on a fractional-like approximation method which makes simultaneous use of power series and asymptotic expansions [2], was recently published; that approximation differs from the usual two-points Pade method [3–5] in that, besides the usual quotient of polynomials, use is made of fractional powers of first-order degree polynomials, and of trigonometrical factors. Although that approximation is very simple and precise (its maximum absolute error is 4×10^{-3}) it does have two singularities in the negative part of the real axis (a ramification point and a pole), thus when such approximation is used under the integral sign, it does not allow, in general, to obtain a formula for the integral; the integration has to be done numerically. For this reason we have applied that same method, to find a new approximation to J_0, where both singularities are confluent to the same point; in this way an exact integration can be performed. The accuracy of this new approximation is just slightly less than the one previously published [1], however the simplification which it provides for the integration of J_0 is remarkable. In fact, the larger the value of the variable, the better the accuracy of our approximation, which means therefore that it can be integrated over infinite intervals. Moreover, our approximation reproduces the zeroes of J_0 with great accuracy. Since the function J_0 often appears in theoretical physics and, in particular, in optics, we think that our approximation will be very useful. We have thus applied it to compute the integral which gives the transmittance through a circular aperture of known radius a, in the so-called Kirchhoff approximation [6]. The expression which is then obtained for the transmittance recovers the whole form of the exact integral and is practically indistinguishable from the exact values of the transmittance function, over the interval of interest of the aperture, when both are plotted with the same scaling. We have thus resorted to plot the absolute error of the approximation after due amplification by three orders of magnitude. Our approximation for the transmittance not only recovers all the fluctuations of the transmittance function, as the aperture radius changes, but the right location of relative maxima and minima as well. The accuracy is better than 0.01 in the useful range of the aperture values.

We have arranged the material of the present work as follows: in Section 2 we explain how we obtained a new and integrable approximation to $J_0(x)$ which has a

0021-9991/87 $3.00

single singularity. Section 3 deals with the application of our new approximation to evaluate the transmittance by a circular aperture of known radius a. We compare the approximation, thus obtained, with the exact values of the transmittance in Section 4 and give a graph of the absolute error. The last section is devoted to conclusions.

2. An Integrable Fractional Approximation to $J_0(x)$

The definite integral of Bessel's J_0 function, in any interval, appears very often in mathematical physics; there does not exist, however, an easily calculable expression for it. The evaluation of such an integral is frequently done through a double integral, since $J_0(x)$ may be defined itself by an integral expression. The definite integral of J_0, as it is well known, can also be evaluated by a series expansion which, although convergent for any value of the variable, shows a very slow convergency for large values of the variable, requiring a large number of the series terms to be added if the absolute error of the computation is to be kept small. There are also asymptotic expansions for J_0 which again are of relative usefulness since the variable then has to be very large. Several approximations for J_0 [7] can also be found, which are dependent upon the interval in which that function is being evaluated. All these computational procedures are lengthy, and even become cumbersome, when the evaluation of the definite integral of J_0 has to be done continuously in a range of values not close to zero.

In this section we resort to a recently published [2] method to obtain a new fractional-like approximation for J_0, but we improve on the method in order to get an integrable approximation, by forcing the singularities to be confluent to a single value of the variable. The method is very accurate for both small and large values of the independent variable, the two ranges of values which are of major concern and, in addition, the approximations based on it are valid in an infinite interval. The new fractional approximation method as described and used in Refs. [1, 2] reminds one of the two-points Pade approximation method [3–5], in the sense that both make use of expansions around zero and at infinity. Actually the new method is not limited to pure fractional approximations, since it resorts to the factorization of trigonometrical function terms, in such a way that the singularities at infinity of both the function and its resulting approximation are alike, in the sense that they are essential singularities. It therefore provides simpler approximations which have good accuracy. However, for the case of the first-order approximation of Bessel's J_0, the method renders an expression [1] with two singularities which lie outside of the region of approximation: these singularities, however, do not hinder the calculations, nonetheless they make integration of the definite integral of J_0 and of any function defined in terms of that integral rather difficult. The latter cases occur often in physics, and in applied physics, a good example being the calculation of the transmittance of a plane electromagnetic wave through a circular aperture [6]. We have reconsidered this problem and modified the fractional approximation method in such a way that all its advantages are preserved, while at the same time making the two singularities a single one and thus allowing the definite integral of J_0 to be computed in an easier way. For all these reasons we predetermined the form of our new approximation as

$$\hat{J}_0(x) = \frac{1}{(1+qx)^{3/2}} [(P_0 + P_1 x) \cos x + (p_0 + p_1 x) \sin x], \tag{2.1}$$

where P_0, P_1, p_0, p_1, and q are numbers to be determined and J_0 denotes our new approximation.

For x small we use the power series for $(1+qx)^{3/2} J_0(x)$ and, after using equivalent expansions for the terms in the numerator of Eq. (2.1), we obtain

$$\begin{aligned}\left(1 + \frac{3}{2} qx + \frac{3}{8} q^2 x^2 + \cdots\right)&\left(1 - \frac{x^2}{2} + \frac{x^4}{64} + \cdots\right) \\ &\cong (P_0 + P_1 x)\left(1 - \frac{x^2}{2} + \cdots\right) + (p_0 + p_1 x)(x + \cdots).\end{aligned} \tag{2.2}$$

For x large we use the asymptotic expansions [8] of

$$q^{3/2}\left(1 + \frac{1}{qx}\right)^{3/2} x^{3/2} J_0(x),$$

and of the numerator of Eq. (2.1). Since in this second case we only need the leading terms, we obtain

$$\begin{aligned}\frac{q^{3/2}}{\pi^{1/2}}&\left\{(\cos x + \sin x)\left[1 + O\left(\frac{1}{x}\right)\right]\right\} \\ &\cong \left(P_1 + \frac{P_0}{x}\right) \cos x + \left(p_1 + \frac{p_0}{x}\right) \sin x.\end{aligned} \tag{2.3}$$

Equating term by term, from the power series approach, Eq. (2.2), we obtain three equations, and from the asymptotic expansion, Eq. (2.3), we obtain two more, which allow us to calculate the values of the constants P_i, p_i, and q. Our approximation to $J_0(x)$ is therefore given as

$$\hat{J}_0(x) = \frac{1}{27\pi^2} \frac{(27\pi^2 + 512x) \cos x + [32(9\pi - 16) + 512x] \sin x}{(1 + (64/9\pi)\, x)^{3/2}} \tag{2.4}$$

In Fig. 1 we have plotted the absolute error of our approximation in the real interval [0, 18]; it may be seen that the maximum absolute departure from the exact values of $J_0(x)$ is just less than 0.013, and it occurs at about $x = 0.9$. In Fig. 1 we have also plotted Bessel's J_0 function just for the sake of a visual reference.

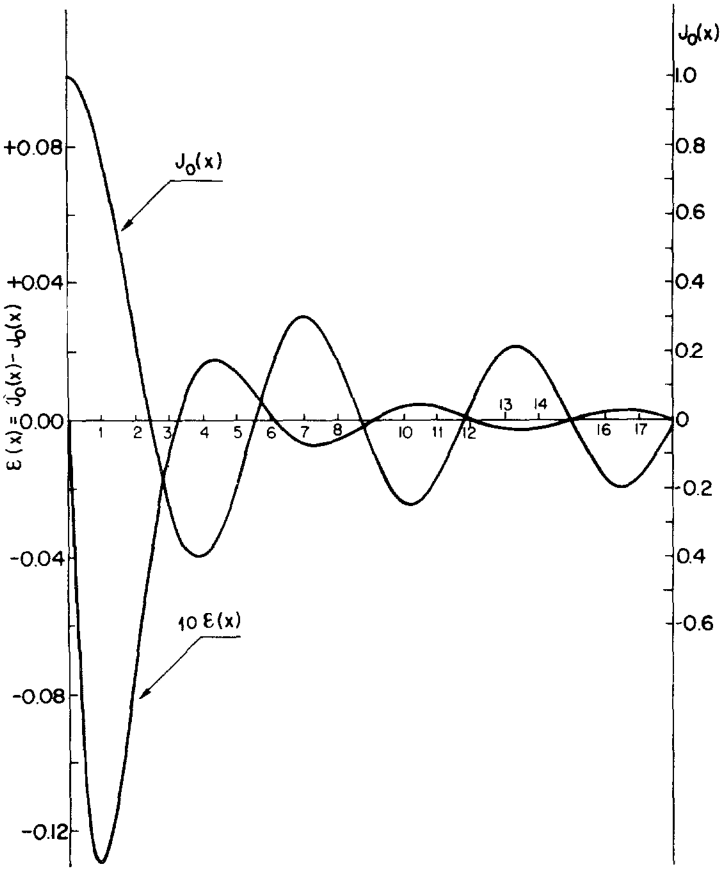

FIG. 1. The absolute error of the approximation, $\varepsilon(x) = \hat{J}_0(x) - J_0(x)$, *enlarged by a factor of 10*, in the (0, 18) interval. The exact function $J_0(x)$ is also plotted as a reference.

3. TRANSMITTANCE THROUGH A CIRCULAR APERTURE

The problem of diffraction of an electro-magnetic wave by a circularly shaped obstacle appears in many relevant studies in optical physics, microwave optics, wave scattering, and radar. One of the parameters of concern is the transmission coefficient, or transmittance T, of a circular aperture in an infinite plane conducting screen [6].

We consider the physical situation of the circular aperture illuminated by normally incident plane wavefronts of wave vector k, s.t. $k = 2\pi/\lambda$, where λ is the light wavelength. It may then be shown [6] that, in the Kirchhoff approximation the transmittance T is given by

$$T(a) = 1 - \frac{1}{2ka} \int_0^{2ka} J_0(t)\, dt, \tag{3.1}$$

where a is the aperture radius.

A plot [9] of the exact values of T, as a function of a, shows that it increases non-monotonically, that it has an oscillating behavior for a, and that it approaches the expected value $T = 1$ for large values of a (Fig. 2). Asymptotic expansions ($ka \gg 1$) for $T(a)$ are given [6] which exhibit the small oscillating behavior explicitly, but they are not very accurate. On the other hand, the power series approach is convergent for finite values but the convergency is very slow, and therefore is not useful for the usual (ka) values. In the present paper we have used the fractional approximation for $J_0(x)$ already obtained (Section 2) in order to find an accurate approximation for T, easily calculable for all values of a. Thus, replacing from Eq. (2.4) in Eq. (3.1) results in

$$T(x) = 1 - \frac{1}{P_0 x} \int_0^x \frac{(P_0 + P_1 t)\cos t + (p_0 + p_1 t)\sin t}{(1 + qt)^{3/2}}\, dt, \tag{3.2}$$

where $x = 2ka$.

After changing the variable to $u = 1 + qt$ and integrating by parts, the latter integral is easily reduced to a sum of the known integrals [10],

$$\int \frac{\sin u, \cos u}{u^s}\, du \qquad \text{with} \quad s = \tfrac{1}{2} \text{ or } \tfrac{3}{2}.$$

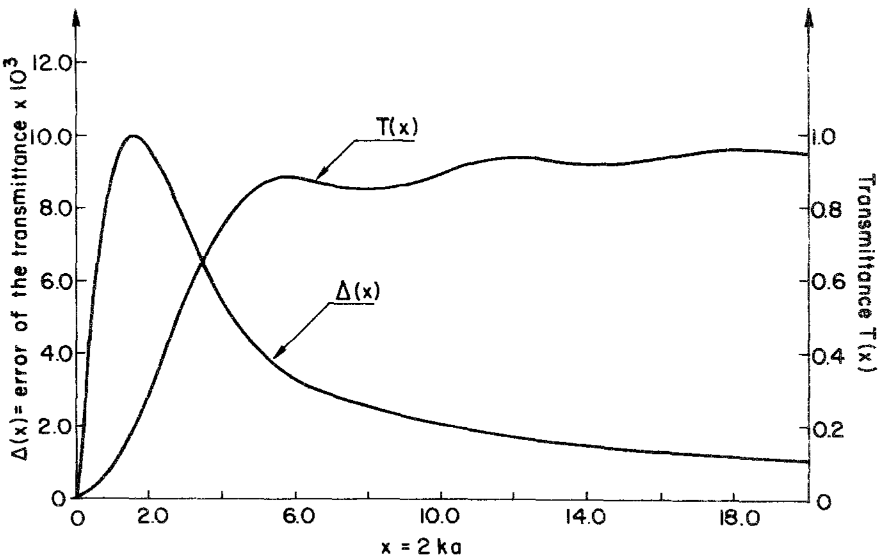

FIG. 2. The absolute error, enlarged by a factor of 10^3, of the transmittance in the interval (0, 20) for $2ka$. The transmittance, in the Kirchhoff approximation, is also plotted as a reference.

These are given in terms of the incomplete gamma function (see Eq. (2.632), p. 183 of [10]), and thus the transmittance can be approximated by

$$T(x)=1-\frac{1}{P_0x}\left\{\frac{(-2)}{q^{3/2}}\left[\frac{P_0\cos x+p_0\sin x}{(b+x)^{1/2}}-\frac{P_0}{b^{1/2}}\right.\right.$$
$$-\left.\left|\frac{m-n}{\sqrt{2}}\operatorname{Re}\Gamma(\tfrac{1}{2},iv)-\frac{n+m}{\sqrt{2}}\operatorname{Im}\Gamma(\tfrac{1}{2},iv)\right|_b^{b+x}\right]$$
$$+\frac{512}{q^{3/2}}\sqrt{2}\,|-\cos b\operatorname{Re}\Gamma(\tfrac{1}{2},iv)+\sin b\operatorname{Im}\Gamma(\tfrac{1}{2},iv)|_b^{b+x}$$
$$+\frac{1024}{q^{5/2}}\left[\frac{\sin x+\cos x}{(b+x)^{1/2}}-\frac{1}{b^{1/2}}\right.$$
$$\left.\left.+\sqrt{2}\,|\sin b\operatorname{Re}\Gamma(\tfrac{1}{2},iv)+\cos b\operatorname{Im}\Gamma(\tfrac{1}{2},iv)|_b^{b+x}\right]\right\}, \qquad (3.3)$$

where $P_0=27\pi^2$ and $q=b^{-1}\doteq 64/(9\pi)$ had been obtained before (Section 2), and the values of the constants m and n are given in the Appendix.

The latter expression, fortunately, can be easily simplified (see the Appendix) to a rather compact and simple equation,

$$T(x)=1-\frac{1}{P_0x}\left|A_1\operatorname{Re}[\Gamma(\tfrac{1}{2},iv)]+A_2\operatorname{Im}[\Gamma(\tfrac{1}{2},iv)]\right.$$
$$\left.-\frac{2}{q^{3/2}}\cdot\frac{B_1\cos(v-b)+B_2\sin(v-b)}{v^{1/2}}\right|_b^{b+x}, \qquad (3.4)$$

where

$$A_1=-276.348987, \qquad A_2=35.672397,$$
$$B_1=40.284648, \qquad B_2=166.584013,$$

are just real numbers (see Appendix).

All that remains, in order to evaluate the transmittance $T(x)$ in Eq. (3.2), is to obtain the values of $\Gamma(\frac{1}{2}, iv)$ in a simple way. A possibility is to resort to well-known mathematical tables, but, in fact, numerical data for the incomplete gamma function is scarce, hence we simply tried to express the latter in terms of a readily evaluable function. Fortunately, using the relation between the incomplete gamma function and the generalized Fresnel integrals (see Eq. (6.5.20), p. 262 of [8]) and performing the integration thus obtained, we were able to show the former function to be given exactly by

$$\Gamma(\tfrac{1}{2},iv)=ie^{-iv}Z[(-iv)^{1/2}], \qquad (3.5)$$

where Z is the well-known plasma dispersion function, i.e., the Hilbert transform of the Gaussian [11]. It happens that Martín *et al.* have developed a procedure to generate a fractional approximation for the Z function [2], which allowed us to find a simple expansion of $Z(s)$ in partial fractions,

$$Z_{ap}(s) = \sum_{k=1}^{n} \left(\frac{a_k}{s - a_k} \right), \tag{3.6}$$

where n is the number of poles of the approximation, and the a_k's and b_k's are the poles and the pole-residues. We have simply used the four-pole expansion for $Z(s)$, taking the poles values from the quoted work [2] and thus completed the calculation of $T(x)$. In passing, we would like to point out that, to the best of our knowledge, the expression for the incomplete gamma function in terms of the Z function in Eq. (3.5) has not been found before and therefore constitutes another original and useful contribution of the present work.

4. Comparison of the Approximated Transmittance with the Exact Values

A very simple computer algorithm, based on a single short loop, allowed the evaluation of $Z(s)$ in Eq. (3.6). It was written as a subroutine of a main program in which the values of $T(x)$, following Eq. (3.4), are calculated (all the computer work was performed with a personal computer). When one plots the exact transmittance and our approximation to it, both graphs are practically superimposed and no differences can be seen (Fig. 2). In order to visualize the differences we have amplified the absolute errors by a factor of 10^3, and this is shown in Fig. 2. It may be seen that our approximate expression for the transmittance is rather accurate; the maximum departure from the exact values is just less than 0.01.

5. Conclusions

We have obtained a very simple fractional approximation for Bessel's J_0 function which facilitates the calculations in which this function is involved, particularly those in which the integral of $J_0(x)$ is required. The new approximation has good accuracy for small and large values of x, the maximum absolute error being 0.013 for $x = 0.9$, and allows a quick evaluation of $J_0(x)$, even with a pocket calculator.

We have applied the approximation to evaluate the transmittance through a circular aperture; the results show that the transmittance values are recovered with great accuracy, the maximum error being less than 0.01 for all useful aperture values. Our approximation allows a quick evaluation of the transmittance, even for large values of the aperture, in comparison with the rather slow computation associated with the power series approximation to J_0.

Appendix

The lengthy expression for the transmittance, Eq. (3.3), may be simplified to the compact one in Eq. (3.4) by first grouping together all the terms in $\mathrm{Re}[\Gamma(\frac{1}{2}, iv)]$ and $\mathrm{Im}[\Gamma(\frac{1}{2}, iv)]$ separately. The constant coefficients for these two then become

$$A_1 = \left(\frac{2}{q^3}\right)^{1/2} \left[(m-n) - 512\cos b + \frac{1024}{q}\sin b\right],$$

$$A_2 = \left(\frac{2}{q^3}\right)^{1/2} \left[-(m+n) + 512\sin b + \frac{1024}{q}\cos b\right],$$

respectively, where $b = q^{-1}$, $m = P_0\cos b - p_0\sin b$, and $n = P_0\sin b + p_0\cos b$.

The remaining variable terms in Eq. (3.3), the ones containing $\cos x$ and $\sin x$, may be rearranged to

$$-\frac{2}{q^{1/2}(b+x)^{1/2}}\left[-\frac{512}{q}(\sin x + \cos x) + (P_0\cos x + p_0\sin x)\right]$$

$$= \frac{-2}{[q^3(b+x)]^{1/2}}(B_1\cos x + B_2\sin x),$$

where

$$B_1 = P_0 - \frac{512}{q}, \qquad B_2 = p_0 - \frac{512}{q}.$$

Finally, the constant terms in Eq. (3.3) may be factored to

$$\frac{2}{(q^3b)^{1/2}}\left(P_0 - \frac{512}{q}\right) = \frac{2}{(q^3b)^{1/2}}B_1,$$

and thus Eq. (3.4) is obtained.

References

1. P. Martín and A. L. Guerrero, *J. Math. Phys.* **26**, 705 (1985).
2. P. Martín, J. Zamudio-Cristi, and G. Donoso, *J. Math. Phys.* **21**, 1332 (1980).
3. W. B. Jones, O. Najstad, and W. J. Thron, *J. Comput. Appl. Math.* **9**, 105 (1983).
4. M. Eiermann, *J. Comput. Appl. Math.* **10**, 219 (1984).
5. W. J. Thron, Two-point Pade tables, *T*-fractions and sequences of Schur, in *Pade and Rational Approximations*, edited by E. B. Saff and R. S. Varga (Academic Press, New York, 1977), p. 215.
6. J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), p. 441.
7. C. A. Wills, J. M. Blair, and P. L. Ragde, *Maths. Comput.* **39**, 617 (1982).
8. M. Abramowitz and I. A. Stegun (Eds.), *Handbook of Mathematical Functions* (Dover, New York, 1970), p. 364.

9. R. W. P. KING AND T. T. WU, *The Scattering and Diffraction of Waves* (Harvard Univ. Press, Cambridge, MA, 1959), Chap. 5, p. 123.
10. I. S. GRADSHTEYN AND I. M. RYZHIK, *Table of Integrals, Series, and Products* (Academic Press, New York, 1980), p. 183.
11. B. D. FRIED AND S. D. CONTE, *The Plasma Dispersión Function* (Academic Press, New York, 1975), p. 1.

RECEIVED: September 26, 1986; REVISED: February 26, 1987

CELSO L. LADERA
PABLO MARTÍN

Departamento de Física,
Universidad Simón Bolívar,
Apartado 80659, Caracas 1081A,
Venezuela